

QUESTIONS ABOUT DETERMINANTS AND POLYNOMIALS

STEVE FISK

This article is an exposition of some questions about determinants and polynomials. We are interested in the following classes of polynomials:

Definition 1.

$\mathcal{U}_d = \left\{ \begin{array}{l} \text{All polynomials } f(x_1, \dots, x_d) \text{ with real coefficients} \\ \text{such that } f(\sigma_1, \dots, \sigma_d) \neq 0 \text{ for all } \sigma_1, \dots, \sigma_d \text{ in the} \\ \text{open upper half plane.} \end{array} \right.$
 $P_d^+ =$ all polynomials in \mathcal{U}_d with all positive coefficients.
 $\mathcal{H}_1 =$ all polynomials with all roots in the closed left half plane.

Polynomials in \mathcal{H}_1 are called *stable polynomials* or sometimes *Hurwitz stable* polynomials. The polynomials in \mathcal{U}_d are called *upper polynomials*. P_1^+ consists of polynomials with all negative roots, and so are also stable. Surveys of these classes of polynomials can be found in [2] and [3]. We say $f(x)$ and $g(x)$ *interlace*, written $f \overset{U}{\leftarrow} g$, if $f + yg \in \mathcal{U}_2$. This is equivalent to the usual definition that the roots of f and g alternate.

Question 1. Suppose that $f = \sum_0^n a_i y^i$ is in P_1^+ . Form the polynomial

$$(1) \quad F(x) = \begin{vmatrix} a_0 & a_1 \\ 0 & a_0 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} x + \cdots + \begin{vmatrix} a_n & 0 \\ a_{n-1} & a_n \end{vmatrix} x^n$$

Show that this is in P_1^+ .

Remark 1. See [4] for some recent results on this problem.

Using the criteria for stability for polynomials of small degree [1] we can show that the polynomials of degree at most 4 are stable. In the case of degree two it follows easily that the polynomials are actually in P_1^+ .

degree = 2: If we write $f(x) = (x + a)(x + b)$ where a, b are positive then

$$F(x) = a^2 b^2 + (a^2 + ba + b^2)x + x^2$$

Since all coefficients are positive $F(x)$ is stable. The discriminant is $(a^2 - ba + b^2)(a^2 + 3ba + b^2)$ which is positive for positive a, b , so F is in P_1^+ .

degree = 3: Write $f(x) = (x + a)(x + b)(x + c)$ with positive coefficients. We find that

$$F(x) = a^2b^2c^2 + (a^2b^2 + a^2bc + ab^2c + a^2c^2 + abc^2 + b^2c^2)x + (a^2 + ab + b^2 + ac + bc + c^2)x^2 + x^3$$

If we write $F = \alpha_0 + \alpha_1x + \alpha_2x^2 + x^3$ then the criterion to be stable is $\alpha_1\alpha_2 - \alpha_0 > 0$. If we compute this expression we get a sum of 19 monomials, all with positive coefficients, so F is stable.

degree = 4: In this case we compute the criterion to be stable, and it is a sum of 201 monomials, all with positive coefficients.

Question 2. Generalize Question 1 by considering

$$T_k(f) = \sum_i \begin{vmatrix} a_i & a_{i+k} \\ a_{i-k} & a_i \end{vmatrix} x^i$$

Show that $T_k(f)$ is in P_1^+ if $f \in P_1^+$.

Remark 2. A computer algebra calculation shows that $T_2(f)$ is stable when f has degree four. If f has degree n and $k > n/2$ then

$$T_k(f) = \sum a_i^2 x^i = f * f$$

where $*$ is the Hadamard product. Thus, $T_k(f) \in P_1^+$ if $k > n/2$.

Computation shows that $T_k(f)$ and $T_j(f)$ do not generally interlace. However, it does appear that there is a $g \in P_1^+$ that interlaces every $T_k(f)$.

Question 3. Choose a positive integer d , and let $\sum_0^n a_i x^i \in P_1^+$. Form

$$(2) \quad F(x) = \sum_i x^i \begin{vmatrix} a_i & \dots & a_{i+d} \\ a_{i-1} & \dots & a_{i+d-1} \\ \vdots & & \vdots \\ a_{i-d} & \dots & a_i \end{vmatrix}$$

Show that $F(x)$ is in P_1^+ .

Remark 3. If $d = 1$ this is just Question 1.

Question 4. If $\sum_0^n a_i x^i \in P_1^+$ then construct the matrix

$$M = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ \vdots & \vdots & & & \end{pmatrix}$$

For any positive integer d construct a new matrix M' by replacing each entry of M by the determinant of the d by d matrix whose upper left corner is that entry. Show that M' is totally positive.

Remark 4. By the Aissen-Schoenberg theorem, if M' is totally positive then the polynomial corresponding to the first row is in P_1^+ . Thus this question implies Questions 1 and 3.

Question 5. Suppose that $f \xleftarrow{U} g$ in P_1^+ . If $f = \sum a_i x^i$, $g = \sum b_i x^i$ then show that

$$\sum_i \begin{vmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{vmatrix} x^i \in P_1^+$$

Remark 5. This implies Question 1. Since $(x+1)f(x) \xleftarrow{U} f(x)$ we have

$$\sum_i \begin{vmatrix} a_i + a_{i-1} & a_{i+1} + a_i \\ a_i & a_{i+1} \end{vmatrix} x^i = \sum_i \begin{vmatrix} a_{i-1} & a_i \\ a_i & a_{i+1} \end{vmatrix} x^i$$

which is Question 1.

Question 6. Suppose $f = \sum a_{i,j} x^i y^j \in P_2^+$. For any positive integer d show that

$$\sum_i x^i \begin{vmatrix} a_{i,0} & \dots & a_{i,d} \\ \vdots & & \vdots \\ a_{i+d,0} & \dots & a_{i+d,d} \end{vmatrix} \in P_1^+$$

Remark 6. Since $f \xleftarrow{U} g$ is equivalent to $f + yg \in U_2$ we see that Question 6 implies Question 5.

Question 7. Consider a polynomial f in P_3^+

$$\begin{aligned} f_{0,0}(x) &+ f_{0,1}(x)y &+ f_{0,2}(x)y^2 &+ \dots \\ f_{1,0}(x)z &+ f_{1,1}(x)yz &+ f_{1,2}(x)y^2z &+ \dots \\ f_{2,0}(x)z^2 &+ f_{2,1}(x)yz^2 &+ f_{2,2}(x)y^2z^2 &+ \dots \end{aligned}$$

We construct a new polynomial by replacing each term by the k by k determinant based at that term. If $k = 2$ then the polynomial is $F(x, y, z) =$

$$\begin{aligned} \begin{vmatrix} f_{0,0} & f_{0,1} \\ f_{1,0} & f_{1,1} \end{vmatrix} &+ \begin{vmatrix} f_{0,1} & f_{0,2} \\ f_{1,1} & f_{1,2} \end{vmatrix} y &+ \begin{vmatrix} f_{0,2} & f_{0,3} \\ f_{1,2} & f_{1,3} \end{vmatrix} y^2 &+ \dots \\ \begin{vmatrix} f_{1,0} & f_{1,1} \\ f_{2,0} & f_{2,1} \end{vmatrix} z &+ \begin{vmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \end{vmatrix} yz &+ \begin{vmatrix} f_{1,2} & f_{1,3} \\ f_{2,2} & f_{2,3} \end{vmatrix} y^2 z &+ \dots \end{aligned}$$

Show that if $k = 2$ then $F(x, \alpha, \beta)$ is stable for positive α, β . In addition, for all k all coefficients have the same sign.

Remark 7. If all $f_{i,j}(x)$ are constant, so that $F(x, y, z) = G(y, z)$ where $G \in P_2^+$, then $G(x, 0)$ is equation (3), and so would be in P_1^+ , rather than just a stable polynomial.

Here are two simple cases. If we take $f = (x + y + z)^2$ then

$$F(x, y, z) = -2(x^2 + y + z)$$

This is not stable, but is stable for positive y, z .

If $f = (x + y + z)^3$ then

$$F(x, y, z) = -3(x^4 + 3yx^2 + 3zx^2 + y^2 + z^2 + 3yz)$$

Again, $F(x, y, z)$ is not stable, and can be checked to be stable for positive y, z .

If we consider $k = 3$ and $f = (x + y + z)^3$ then $F(x, y, z) = -9(x^3 + y + z)$ which is not a stable polynomial and $F(x, \alpha, \beta) \notin P_1^+$ for all positive α, β , but it does have all negative coefficients.

Question 8. Suppose that $f \in P_1^+$. Show that the determinant of the matrix below is stable. We know that it is positive for positive x .

$$(3) \quad F(x) = \begin{pmatrix} f & f' & \dots & f^{(d)} \\ \vdots & \vdots & & \vdots \\ f^{(d)} & f^{(d+1)} & \dots & f^{(2d)} \end{pmatrix}$$

Question 9. Suppose $f \in P_2^+$, and write $f = \sum f_i(x)y^i$. Consider the polynomial

$$(4) \quad F(x) = \begin{vmatrix} f_i & f_{i+1} & \dots & f_{i+d} \\ f_{i+1} & f_{i+2} & \dots & f_{i+d+1} \\ \vdots & \vdots & & \vdots \\ f_{i+d} & f_{i+d+1} & \dots & f_{2d} \end{vmatrix}$$

Show that $F(x)$ is stable for all positive integers d and non-negative integers i .

Remark 8. We know that this holds for $d = 1$. If we consider $i = 0$ and $f(x + y)$ where $f \in P_1^+$ then from (4) we have an assertion that appears to be stronger than Question 8:

$$\begin{vmatrix} f & f' & f^{(2)}/2! & \dots & f^{(d)}/d! \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f^{(d)}/d! & f^{(d+1)}/(d+1)! & \dots & \dots & f^{(2d)}/(2d)! \end{vmatrix} \quad \text{is stable.}$$

Question 10. Suppose $f \in P_2^+$, and write $f = \sum f_i(x)y^i$. Form the infinite matrix

$$(5) \quad \begin{pmatrix} f_0 & f_1 & f_2 & f_3 & \dots \\ 0 & f_0 & f_1 & f_2 & \dots \\ 0 & 0 & f_0 & f_1 & \dots \\ 0 & 0 & 0 & f_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Show that the resulting matrix is totally stable. That is, every minor is a stable polynomial.

Remark 9. If we substitute a positive value for x then the resulting matrix is well known to be totally positive. Question 9 is a consequence of this question since the determinant in (4) is a minor of (5).

For example, if $f = (x + y)^n$ then it appears that all minors of (5) have the form cx^s for positive c and non-negative integer s .

Question 11. Suppose that $f = \prod (x + b_i y + c_i)$ where b_i, c_i are positive, and write $f = \sum a_{ij} x^i y^j$. Consider the matrix

$$\begin{pmatrix} a_{0d} & \cdots & a_{00} \\ \vdots & & \vdots \\ a_{dd} & \cdots & a_{d0} \end{pmatrix}$$

Show that the matrix is totally positive for any positive integer d .

Remark 10. f is in P_2^+ , but the assertion fails for arbitrary polynomials in P_2^+ . However, since consecutive rows are the coefficients of interlacing polynomials, all two by two determinants are positive for any $f \in P_2^+$.

If we take $\prod_1^3 (x + b_i y + c_i)$ and $d = 2$ then the determinant is a sum of 7 monomials with all positive coefficients.

Here's an example where it fails for $f \in P_2^+$.

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + x \begin{pmatrix} 13 & 9 & 7 \\ 9 & 7 & 5 \\ 7 & 5 & 4 \end{pmatrix} + y \begin{pmatrix} 5 & 7 & 8 \\ 7 & 11 & 12 \\ 8 & 12 & 14 \end{pmatrix}$$

The three matrices are positive definite, so the determinant of M is in P_2^+ , and equals

$$1 + 24x + 16x^2 + 2x^3 + 30y + 164xy + 62x^2y + 22y^2 + 64xy^2 + 4y^3$$

with coefficient matrix

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 22 & 64 & 0 & 0 \\ 30 & 164 & 62 & 0 \\ 1 & 24 & 16 & 2 \end{pmatrix}$$

The determinant of the three by three matrix in the lower left corner is -1760 . Of course, all the two by two submatrices have positive determinant.

Question 12. When is the product of two totally stable matrices a totally stable matrix?

Question 13. A totally upper matrix has the property that every minor is either zero or a polynomial with all real roots.

- (1) What are constructions of totally upper matrices and totally stable matrices?
- (2) When is the product of two totally upper matrices a totally upper matrix?

REFERENCES

- [1] Steve Fisk, *Polynomials, roots, and interlacing*, available at [arXiv:math.CA/0612833](https://arxiv.org/abs/math/0612833).
- [2] ———, *An introduction to upper half plane polynomials*, available at [arxiv:math.CA/0711.4043](https://arxiv.org/abs/math/0711.4043).
- [3] ———, *Aspects of stable polynomials*, available at [arxiv:math.CA/0803.0286](https://arxiv.org/abs/math/0803.0286).
- [4] Bruce E. Sagan Peter R. W. McNamara, *Infinite log-concavity: developments and conjectures*, available at [arxiv:math.CO/0808.1065v1](https://arxiv.org/abs/math/0808.1065v1).